

NS-NS Sector of Closed Superstring Field Theory

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Abstract

We give a construction for a general class of vertices in superstring field theory which include integration over bosonic moduli as well as the required picture changing insertions. We apply this procedure to find a covariant action for the NS-NS sector of Type II closed superstring field theory.

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1 Introduction

Though bosonic string field theory has been well-understood since the mid 90's [1, 2, 3, 4], superstring field theory remains largely mysterious. In some cases it is possible to find elegant formulations utilizing the large Hilbert space [5, 6, 7, 8, 9], but it seems difficult to push beyond tree level [10, 11, 12, 13] and the presumed geometrical underpinning of the theory in terms of the supermoduli space remains obscure. A somewhat old-fashioned alternative [14] is to formulate superstring field theory using fields in the small Hilbert space. A well known complication, however, is that one needs a prescription for inserting picture changing operators into the action. This requires an apparently endless sequence of choices, and while limited work in this direction exists [15, 16, 17], it has not produced a compelling and fully explicit action.

Recent progress on this problem for the open superstring was reported in [18], inspired by studies of gauge fixing in Berkovits' open superstring field theory [19]. The basic insight of [18] is that the multi-string products of open superstring field theory can be constructed by passing to the large Hilbert space and constructing a particular finite gauge transformation through the space of A_∞ structures. The result is an explicit action for open superstring field theory which automatically satisfies the classical BV master equation. In this paper we generalize these results to define classical actions for the NS sectors of all open and closed superstring field theories. Of particular interest is the NS-NS sector of Type II closed superstring field theory, for which a construction in the

large Hilbert space appears difficult [20].⁴ The main technical obstacle for us will be learning how to accommodate vertices which include integration over bosonic moduli, and for the NS-NS superstring, how to insert additional picture changing operators for the rightmoving sector. These results lay the groundwork for serious consideration of the Ramond sector and quantization of superstring field theory. This is of particular interest in the context of recent efforts to obtain a more complete understanding of superstring perturbation theory [22, 23, 24, 25, 26].

This paper is organized as follows. In section 2 we review the algebraic formulation of open and closed string field theory in terms of A_∞ and L_∞ algebras, with an emphasis on the coalgebra description. This mathematical language gives a compact and convenient notation for expressing various multi-string products and their interrelation. In section 3 we revisit Witten's open superstring field theory in the -1 picture [14], but generalizing [18], we allow vertices which include integration over bosonic moduli as well as the required picture changing insertions. We find that the multi-string products can be derived from a recursion involving a two-dimensional array of products of intermediate picture number. The recursion emerges from the solution to a pair of differential equations which follow uniquely from two assumptions: that the products are derived by gauge transformation through the space of A_∞ structures, and that the gauge transformation is defined in the large Hilbert space. In section 4, we explain how this construction generalizes (with little effort) to the NS sector of heterotic string field theory. In section 5 we consider the NS-NS sector of Type II closed superstring field theory. We give one construction which defines the products by applying the open string recursion of section 3 twice, first to get the correct picture in the leftmoving sector and again to get the correct picture in the rightmoving sector. This construction however treats the left and rightmoving sectors asymmetrically. We therefore provide a second, more nontrivial construction which preserves symmetry between left and rightmovers at every stage in the recursion. We end with some conclusions.

2 A_∞ and L_∞ Algebras

Here we review the algebraic formulation of open and closed string field theory in the language on A_∞ and L_∞ algebras. For the A_∞ case the discussion basically repeats section 4 of [18]. For more mathematical discussion see [27] for A_∞ and [28, 29] for L_∞ .

2.1 A_∞

Let's start with the A_∞ case. Here the basic objects are multi-products b_n on a \mathbb{Z}_2 -graded vector space \mathcal{H}

$$b_n(\Psi_1, \dots, \Psi_n) \in \mathcal{H}, \quad \Psi_i \in \mathcal{H}, \quad (2.1)$$

⁴A recent proposal for Type II closed superstring field theory in the large Hilbert space appears in [21]. Interestingly, however, picture changing operators still appear to be needed in the action.

which have no particular symmetry upon interchange of the arguments. For us, \mathcal{H} is the open string state space and the \mathbb{Z}_2 grading, called *degree*, is Grassmann parity plus one. The product b_n defines a linear map from the n -fold tensor product of \mathcal{H} into \mathcal{H} :

$$b_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}. \quad (2.2)$$

If we have a state in $\mathcal{H}^{\otimes n}$ of the form

$$\Psi_1 \otimes \Psi_2 \otimes \dots \otimes \Psi_n \in \mathcal{H}^{\otimes n}, \quad (2.3)$$

then b_n acts on such a state as

$$b_n(\Psi_1 \otimes \dots \otimes \Psi_n) = b_n(\Psi_1, \dots, \Psi_n), \quad (2.4)$$

where the right hand side is the multi-string product as written in (2.1). Since the states (2.3) form a basis, this equation defines the action of b_n on the whole tensor product space.

Now suppose we have two linear maps

$$\begin{aligned} A : \mathcal{H}^{\otimes k} &\rightarrow \mathcal{H}^{\otimes l}, \\ B : \mathcal{H}^{\otimes m} &\rightarrow \mathcal{H}^{\otimes n}. \end{aligned} \quad (2.5)$$

We will find it useful to define the tensor product map:

$$A \otimes B : \mathcal{H}^{\otimes k+m} \rightarrow \mathcal{H}^{\otimes l+n}. \quad (2.6)$$

Applying this to a state of the form (2.3) gives

$$\begin{aligned} A \otimes B(\Psi_1 \otimes \dots \otimes \Psi_k \otimes \Psi_{k+1} \otimes \dots \otimes \Psi_{k+m}) &= (-1)^{\deg B(\deg \Psi_1 + \dots + \deg \Psi_k)} \\ &\quad \times A(\Psi_1 \otimes \dots \otimes \Psi_k) \otimes B(\Psi_{k+1} \otimes \dots \otimes \Psi_{k+m}). \end{aligned} \quad (2.7)$$

There may be a sign from commuting B past the first k states. We are particularly interested in tensor products of b_n with the identity map on \mathcal{H} , which we denote \mathbb{I} .

With these preparations, we can define a natural action of the n -string product b_n on the tensor algebra of \mathcal{H} :⁵

$$\mathbf{b}_n : T\mathcal{H} \rightarrow T\mathcal{H}, \quad T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \dots. \quad (2.8)$$

The tensor algebra has a natural coalgebra structure, on which \mathbf{b}_n acts as a coderivation (see, for example, [27]). We will usually indicate the coderivation corresponding to an

⁵In our case, we should identify $\mathcal{H}^{\otimes 0} = \mathbb{C}$.

n -string product with boldface. The coderivation \mathbf{b}_n can be defined by its action on each $\mathcal{H}^{\otimes N}$ component of the tensor algebra. If it acts on $\mathcal{H}^{\otimes N \geq n}$, we have

$$\mathbf{b}_n \Psi \equiv \sum_{k=0}^{N-n} \mathbb{I}^{\otimes N-n-k} \otimes b_n \otimes \mathbb{I}^{\otimes k} \Psi, \quad \Psi \in \mathcal{H}^{\otimes N \geq n} \subset T\mathcal{H}. \quad (2.9)$$

If it acts on the $\mathcal{H}^{\otimes N < n}$ component, by definition \mathbf{b}_n vanishes. One can check that the commutator⁶ of two coderivations is also a coderivation. For example, if \mathbf{b}_m and \mathbf{c}_n are coderivations derived from the multi-string products

$$\begin{aligned} b_m : \mathcal{H}^{\otimes m} &\rightarrow \mathcal{H}, \\ c_n : \mathcal{H}^{\otimes n} &\rightarrow \mathcal{H}, \end{aligned} \quad (2.10)$$

then the commutator $[\mathbf{b}_m, \mathbf{c}_n]$ is a coderivation derived from the $m+n-1$ string product,

$$[b_m, c_n] \equiv b_m \sum_{k=0}^{m-1} \mathbb{I}^{\otimes m-1-k} \otimes c_n \otimes \mathbb{I}^{\otimes k} - (-1)^{\deg(b_m)\deg(c_n)} c_n \sum_{k=0}^{n-1} \mathbb{I}^{\otimes n-1-k} \otimes b_m \otimes \mathbb{I}^{\otimes k}. \quad (2.11)$$

This means that multi-string products in open string field theory, packaged in the form of coderivations, naturally define a graded Lie algebra. This fact is very useful for simplifying the expression of the A_∞ relations.

Open string field theory is defined by a sequence of multi-string products of odd degree satisfying the relations of a cyclic A_∞ algebra. We denote these products

$$M_1 = Q, \ M_2, \ M_3, \ M_4, \ \dots, \quad (2.12)$$

where Q is the BRST operator and

$$M_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}. \quad (2.13)$$

The A_∞ relations imply that the BRST variation of the n th product M_n is related to sums of compositions of lower products $M_{k < n}$. This is most conveniently expressed using coderivations:

$$[\mathbf{M}_1, \mathbf{M}_n] + [\mathbf{M}_2, \mathbf{M}_{n-1}] + \dots + [\mathbf{M}_{n-1}, \mathbf{M}_2] + [\mathbf{M}_n, \mathbf{M}_1] = 0. \quad (2.14)$$

The first and last terms represent the BRST variation of M_n . For example, the fact that Q is a derivation of the 2-product is expressed by the equation,

$$[Q, M_2] = 0. \quad (2.15)$$

Using (2.11), this implies

$$QM_2 + M_2(Q \otimes \mathbb{I} + \mathbb{I} \otimes Q) = 0, \quad (2.16)$$

⁶Commutators in this paper are always graded with respect to degree.

and acting on a pair of states $\Psi_1 \otimes \Psi_2$ gives

$$QM_2(\Psi_1, \Psi_2) + M_2(Q\Psi_1, \Psi_2) + (-1)^{\deg(\Psi_1)}M_2(\Psi_1, Q\Psi_2) = 0, \quad (2.17)$$

which is the familiar expression of the fact that Q is a derivation (recalling that M_2 has odd degree.) To write the action, we need one more ingredient: a symplectic form

$$\langle \omega | : \mathcal{H}^{\otimes 2} \rightarrow \mathbb{C}. \quad (2.18)$$

Writing $\langle \omega | \Psi_1 \otimes \Psi_2 = \omega(\Psi_1, \Psi_2)$, the symplectic form is related to the BPZ inner product through

$$\omega(\Psi_1, \Psi_2) = (-1)^{\deg(\Psi_1)} \langle \Psi_1, \Psi_2 \rangle, \quad (2.19)$$

and is graded antisymmetric:

$$\omega(\Psi_1, \Psi_2) = -(-1)^{\deg(\Psi_1)\deg(\Psi_2)}\omega(\Psi_2, \Psi_1). \quad (2.20)$$

Gauge invariance requires that n -string products are BPZ odd:

$$\langle \omega | \mathbb{I} \otimes M_n = -\langle \omega | M_n \otimes \mathbb{I}, \quad (2.21)$$

so that they give rise to cyclic vertices (in this case the products define a so-called *cyclic* A_∞ algebra). Then we can write a gauge invariant action

$$S = \sum_{n=0}^{\infty} \frac{1}{n+2} \omega(\Psi, M_{n+1}(\underbrace{\Psi, \dots, \Psi}_{n+1 \text{ times}})). \quad (2.22)$$

2.2 L_∞

Now let's discuss the L_∞ case. The basic objects are multi-products b_n on a \mathbb{Z}_2 -graded vector space \mathcal{H}

$$b_n(\Phi_1, \dots, \Phi_n) \in \mathcal{H}, \quad \Phi_i \in \mathcal{H}, \quad (2.23)$$

which are *graded symmetric* upon interchange of the arguments. For us, \mathcal{H} is the closed string state space, and the \mathbb{Z}_2 grading, called *degree*, is identical to Grassmann parity (unlike for the open string, where degree is identified with Grassmann parity plus one.). Since the products are (graded) symmetric upon interchange of inputs, they naturally act on a symmetrized tensor algebra. We will denote the symmetrized tensor product with a wedge \wedge . It satisfies

$$\Phi_1 \wedge \Phi_2 = (-1)^{\deg(\Phi_1)\deg(\Phi_2)}\Phi_2 \wedge \Phi_1, \quad \Phi_1 \wedge (\Phi_2 \wedge \Phi_3) = (\Phi_1 \wedge \Phi_2) \wedge \Phi_3. \quad (2.24)$$

The wedge product is related to the tensor product through the formula

$$\Phi_1 \wedge \Phi_2 \wedge \dots \wedge \Phi_n = \sum_{\sigma} (-1)^{\epsilon(\sigma)} \Phi_{\sigma(1)} \otimes \Phi_{\sigma(2)} \otimes \dots \otimes \Phi_{\sigma(n)}, \quad \Phi_i \in \mathcal{H}. \quad (2.25)$$

The sum is over all distinct permutations σ of $1, \dots, n$, and the sign $(-1)^{\epsilon(\sigma)}$ is the obvious sign obtained by moving $\Phi_1, \Phi_2, \dots, \Phi_n$ past each other into the order prescribed by σ . Note that if some of the factors in the wedge product are the identical, some permutations in the sum may produce an identical term, which effectively produces a $k!$ for k degree even identical factors (degree odd identical factors vanish when taking the wedge product). With these definitions, the closed string product b_n can be seen as a linear map from the n -fold wedge product of \mathcal{H} into \mathcal{H} :

$$b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}. \quad (2.26)$$

Acting on a state of the form (2.25),

$$b_n(\Phi_1 \wedge \dots \wedge \Phi_n) = b_n(\Phi_1, \dots, \Phi_n), \quad (2.27)$$

where the right hand side is the n string product as denoted in (2.23). Since the states (2.25) form a basis, this defines the action of b_n on all states in $\mathcal{H}^{\wedge n}$.

We can define the wedge product between linear maps in a similar way as between states: We replace wedge products with tensor products and sum over permutations, as in (2.25). Therefore, the wedge product of linear maps is implicitly defined by the tensor product of linear maps, via (2.7). While this seems natural, expanding multiple wedge products out into tensor products is usually cumbersome. However, the net result is simple. Suppose we have two linear maps between symmetrized tensor products of \mathcal{H} :

$$\begin{aligned} A : \mathcal{H}^{\wedge k} &\rightarrow \mathcal{H}^{\wedge l}, \\ B : \mathcal{H}^{\wedge m} &\rightarrow \mathcal{H}^{\wedge n}. \end{aligned} \quad (2.28)$$

Their wedge product defines a map

$$A \wedge B : \mathcal{H}^{\wedge k+m} \rightarrow \mathcal{H}^{\wedge l+n}. \quad (2.29)$$

On states of the form (2.25), $A \wedge B$ acts as

$$A \wedge B(\Phi_1 \wedge \Phi_2 \wedge \dots \wedge \Phi_{k+m}) = \sum_{\sigma}' (-1)^{\epsilon(\sigma)} A(\Phi_{\sigma(1)} \wedge \dots \wedge \Phi_{\sigma(k)}) \wedge B(\Phi_{\sigma(k+1)} \wedge \dots \wedge \Phi_{\sigma(k+m)}), \quad (2.30)$$

where σ is a permutation of $1, \dots, k+m$, and Σ' means that we sum only over permutations which change the inputs of A and B . (Permutations which only move around inputs of A and B produce the same terms, and are only counted once). The sign $\epsilon(\sigma)$ is the sign obtained from moving the Φ_i s past each other and past B to obtain the ordering required by σ . For example, let's consider wedge products of the identity map, where potentially confusing symmetry factors arise. Act $\mathbb{I} \wedge \mathbb{I}$ on a pair of states using (2.30):

$$\begin{aligned} \mathbb{I} \wedge \mathbb{I}(\Phi_1 \wedge \Phi_2) &= \mathbb{I}(\Phi_1) \wedge \mathbb{I}(\Phi_2) + (-1)^{\deg(\Phi_1)\deg(\Phi_2)} \mathbb{I}(\Phi_2) \wedge \mathbb{I}(\Phi_1), \\ &= 2\Phi_1 \wedge \Phi_2. \end{aligned} \quad (2.31)$$

Here we find a factor of two because there are two permutations of Φ_1, Φ_2 which switch entries between the first and second maps. Alternatively, we can compute this by expanding in tensor products:

$$\begin{aligned}\mathbb{I} \wedge \mathbb{I}(\Phi_1 \wedge \Phi_2) &= (\mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I}) \left(\Phi_1 \otimes \Phi_2 + (-1)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \otimes \Phi_1 \right), \\ &= 2(\Phi_1 \otimes \Phi_2 + (-1)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \otimes \Phi_1), \\ &= 2\Phi_1 \wedge \Phi_2.\end{aligned}\tag{2.32}$$

Here the factor of two comes because there are two ways to arrange the first and second identity map (which happen to be identical). In this way, it is easy to see that the identity operator on $\mathcal{H}^{\wedge n}$ is given by

$$\mathbb{I}_n \equiv \frac{1}{n!} \underbrace{\mathbb{I} \wedge \dots \wedge \mathbb{I}}_{n \text{ times}} = \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{n \text{ times}}.\tag{2.33}$$

The inverse factor of $n!$ is needed to cancel the $n!$ over-counting of identical permutations of \mathbb{I} .

With these preparations, we can lift the closed string product b_n to a coderivation on the symmetrized tensor algebra:⁷

$$\mathbf{b}_n : S\mathcal{H} \rightarrow S\mathcal{H}, \quad S\mathcal{H} = \mathcal{H}^{\wedge 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\wedge 2} \oplus \dots.\tag{2.34}$$

On the $\mathcal{H}^{\wedge N \geq n}$ component of the symmetrized tensor algebra, \mathbf{b}_n acts as

$$\mathbf{b}_n \Phi \equiv (b_n \wedge \mathbb{I}_{N-n})\Phi, \quad \Phi \in \mathcal{H}^{\wedge N \geq n} \subset S\mathcal{H},\tag{2.35}$$

and on the $\mathcal{H}^{\wedge N < n}$ component \mathbf{b}_n vanishes. If \mathbf{b}_m and \mathbf{c}_n are coderivations derived from the products

$$\begin{aligned}b_m : \mathcal{H}^{\wedge m} &\rightarrow \mathcal{H}, \\ c_n : \mathcal{H}^{\wedge n} &\rightarrow \mathcal{H},\end{aligned}\tag{2.36}$$

then the commutator $[\mathbf{b}_m, \mathbf{c}_n]$ is a coderivation derived from the $m+n-1$ -string product,

$$[b_m, c_n] \equiv b_m(c_n \wedge \mathbb{I}_{m-1}) - (-1)^{\deg(b_m)\deg(c_n)} c_n(b_m \wedge \mathbb{I}_{n-1}).\tag{2.37}$$

This means that, when described as coderivations on the symmetrized tensor algebra, the products of closed string field theory naturally define a graded Lie algebra.

Closed string field theory is defined by a sequence of multi-string products of odd degree satisfying the relations of a cyclic L_∞ algebra. We denote these products

$$L_1 = Q, \ L_2, \ L_3, \ L_4, \ \dots,\tag{2.38}$$

⁷We identify $\mathcal{H}^{\wedge 0} = \mathbb{C}$.

where

$$L_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}. \quad (2.39)$$

The L_∞ relations imply that the BRST variation of the n th closed string product L_n is related to sums of compositions of lower products $L_{k < n}$. In fact, expressed using coderivations, the L_∞ relations have the same formal structure as the A_∞ relations:

$$[\mathbf{L}_1, \mathbf{L}_n] + [\mathbf{L}_2, \mathbf{L}_{n-1}] + \dots + [\mathbf{L}_{n-1}, \mathbf{L}_2] + [\mathbf{L}_n, \mathbf{L}_1] = 0. \quad (2.40)$$

What makes these relations different is the \mathbf{L}_n s act on the symmetrized tensor algebra, rather than the tensor algebra as for the open string. Consider for example the third L_∞ relation,

$$[\mathbf{Q}, \mathbf{L}_3] + \frac{1}{2}[\mathbf{L}_2, \mathbf{L}_2] = 0, \quad (2.41)$$

which should characterize the failure of the Jacobi identity for L_2 in terms of the BRST variation of L_3 . To write this identity directly in terms of the products, use (2.37):

$$QL_3 + L_3(Q \wedge \mathbb{I}_2) + L_2(L_2 \wedge \mathbb{I}) = 0. \quad (2.42)$$

Acting on a wedge product of three states, according to (2.30) we must sum over distinct permutations of the states on the inputs. With (2.27), this gives a somewhat lengthy expression:

$$\begin{aligned} 0 = & QL_3(\Phi_1, \Phi_2, \Phi_3) + L_3(Q\Phi_1, \Phi_2, \Phi_3) + (-1)^{\deg(\Phi_1)(\deg(\Phi_2)+\deg(\Phi_3))} L_3(Q\Phi_2, \Phi_3, \Phi_1) \\ & + (-1)^{\deg(\Phi_3)(\deg(\Phi_1)+\deg(\Phi_2))} L_3(Q\Phi_3, \Phi_1, \Phi_2) \\ & + L_2(L_2(\Phi_1, \Phi_2), \Phi_3) + (-1)^{\deg(\Phi_3)(\deg(\Phi_1)+\deg(\Phi_2))} L_2(L_2(\Phi_3, \Phi_1), \Phi_2) \\ & + (-1)^{\deg(\Phi_1)(\deg(\Phi_2)+\deg(\Phi_3))} L_2(L_2(\Phi_2, \Phi_3), \Phi_1). \end{aligned} \quad (2.43)$$

The first four terms represent the BRST variation of L_3 , and the last three terms represent the Jacobiator computed from L_2 .

To write the action, we need a symplectic form for closed strings:

$$\langle \omega | : \mathcal{H}^{\otimes 2} \rightarrow \mathbb{C}. \quad (2.44)$$

Note that $\langle \omega |$ acts on a tensor product of two closed string states (rather than the wedge product, which would vanish by symmetry). Writing $\langle \omega | \Phi_1 \otimes \Phi_2 = \omega(\Phi_1, \Phi_2)$, the symplectic form is related to the closed string inner product through

$$\omega(\Phi_1, \Phi_2) = (-1)^{\deg(\Phi_1)} \langle \Phi_1, c_0^- \Phi_2 \rangle, \quad (2.45)$$

where $c_0^- \equiv c_0 - \bar{c}_0$.⁸ Closed string fields are assumed to satisfy the constraints

$$\begin{aligned} b_0^- \Phi &= 0, & b_0^- &\equiv b_0 - \bar{b}_0, \\ L_0^- \Phi &= 0, & L_0^- &\equiv L_0 - \bar{L}_0. \end{aligned} \quad (2.47)$$

With these conventions the symplectic form is graded antisymmetric:⁹

$$\omega(\Phi_1, \Phi_2) = -(-1)^{\deg(\Phi_1)\deg(\Phi_2)} \omega(\Phi_2, \Phi_1). \quad (2.48)$$

Gauge invariance requires that n -string products are BPZ odd:

$$\langle \omega | \mathbb{I} \otimes L_n = -\langle \omega | L_n \otimes \mathbb{I}. \quad (2.49)$$

This implies that the vertices are symmetric under permutations of the inputs. (This is called a *cyclic* L_∞ algebra, though the vertices have full permutation symmetry). With these ingredients, we can write a gauge invariant closed string action,

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \omega(\Phi, L_{n+1}(\underbrace{\Phi, \dots, \Phi}_{n+1 \text{ times}})). \quad (2.50)$$

3 Witten's Theory with Stubs

In this section we revisit the construction of Witten's open superstring field theory. Unlike [18], where the higher vertices were built from Witten's open string star product, here we consider a more general set of vertices which may include integration over bosonic moduli. Such vertices are at any rate necessary for the closed string [30].

Witten's superstring field theory is based on a string field Ψ in the -1 picture. It has even degree (but is Grassmann odd), ghost number 1, and lives in the small Hilbert space. The action is defined by a sequence of multi-string products

$$M_1^{(0)} = Q, \quad M_2^{(1)}, \quad M_3^{(2)}, \quad M_4^{(3)}, \quad \dots, \quad (3.1)$$

satisfying the relations of a cyclic A_∞ algebra. Since the vertices must have total picture -2 , and the string field has picture -1 , the $(n+1)$ st product $M_{n+1}^{(n)}$ must carry picture

⁸The BPZ inner product

$$\langle \Phi_1, \Phi_2 \rangle = \langle I \circ \mathcal{V}_{\Phi_1}(0) \mathcal{V}_{\Phi_2}(0) \rangle \quad (2.46)$$

is conventionally defined with the conformal map $I(z) = 1/z$ for closed strings.

⁹The extra sign in front of the closed string inner product in (2.45) was chosen to ensure graded antisymmetry of the symplectic form. Without the sign, the closed string inner product itself has the symmetry of an odd symplectic form, like the antibracket. This symmetry however is somewhat awkward to describe in the tensor algebra language. Note that, with our choice of symplectic form, permutation symmetry of the vertices produces signs from moving fields through the products L_n .

n .¹⁰ We keep track of the picture through the upper index of the product. The goal is to construct these products by placing picture changing operators on a set of n -string products defining open bosonic string field theory:

$$M_1^{(0)} = Q, \quad M_2^{(0)}, \quad M_3^{(0)}, \quad M_4^{(0)}, \quad \dots, \quad (3.2)$$

where the bosonic string products of course carry zero picture. We can choose $M_2^{(0)}$ to be Witten's open string star product, in which case the higher bosonic products $M_3^{(0)}, M_4^{(0)}, \dots$ can be chosen to vanish. This is the scenario considered in [18]. Here we will not assume that $M_3^{(0)}, M_4^{(0)}, \dots$ vanish. For example, we can consider the open string star product with “stubs” attached to each output:

$$M_2^{(0)}(A, B) = (-1)^{\deg(A)} e^{-\pi L_0} \left((e^{-\pi L_0} A) * (e^{-\pi L_0} B) \right). \quad (3.3)$$

The presence of stubs means that the propagators by themselves will not cover the full bosonic moduli space, and the higher products $M_3^{(0)}, M_4^{(0)}, \dots$ are needed to cover the missing regions. Though it is natural to think of the $M_n^{(0)}$'s as deriving from open bosonic string field theory, this is not strictly necessary. We only require three formal properties:

- 1) The $M_n^{(0)}$'s satisfy the relations of a cyclic A_∞ algebra.
- 2) The $M_n^{(0)}$'s are in the small Hilbert space.
- 3) The $M_n^{(0)}$'s carry vanishing picture number.

Our task is to add picture number to the $M_n^{(0)}$'s to define consistent nonzero vertices for Witten's open superstring field theory.

3.1 Cubic and Quartic Vertices

We start with the cubic vertex, defined by a 2-product $M_2^{(1)}$ constructed by placing a picture changing operator X once on each output of $M_2^{(0)}$:

$$M_2^{(1)}(\Psi_1, \Psi_2) \equiv \frac{1}{3} \left(X M_2^{(0)}(\Psi_1, \Psi_2) + M_2^{(0)}(X \Psi_1, \Psi_2) + M_2^{(0)}(\Psi_1, X \Psi_2) \right). \quad (3.4)$$

The picture changing operator X takes the following form:

$$X \equiv \oint_{|z|=1} \frac{dz}{2\pi i} f(z) X(z), \quad X(z) = Q\xi(z), \quad (3.5)$$

¹⁰Ghost number saturation is also important, but is essentially automatic in our construction. Suffice it to say that products $M_n^{(k)}$ carry ghost number $2 - n$ and gauge products $\mu_n^{(k)}$ carry ghost number $1 - n$ for the open string. For the closed string, products $L_n^{(p,q)}$ carry ghost number $3 - 2n$ and gauge products $\lambda_n^{(p,q)}, \bar{\lambda}_n^{(p,q)}$ carry ghost number $2 - 2n$.

where $f(z)$ a 1-form which is analytic in some nondegenerate annulus around the unit circle, and satisfies

$$f(z) = -\frac{1}{z^2}f\left(-\frac{1}{z}\right), \quad \oint_{|z|=1} \frac{dz}{2\pi i} f(z) = 1. \quad (3.6)$$

The first relation implies that X is BPZ even, and the second amounts to a choice of the open string coupling constant, which we have set to 1. Since Q and X commute, Q is a derivation of $M_2^{(1)}$:

$$[\mathbf{Q}, \mathbf{M}_2^{(1)}] = 0. \quad (3.7)$$

Together with $[\mathbf{Q}, \mathbf{Q}] = 0$, this means that the first two A_∞ relations are satisfied. However, $M_2^{(1)}$ is not associative, so higher products $M_3^{(2)}, M_4^{(3)}, \dots$ are needed to have a consistent A_∞ algebra.

To find the higher products, the key observation is that $M_2^{(1)}$ is BRST exact in the large Hilbert space:¹¹

$$\mathbf{M}_2^{(1)} = [\mathbf{Q}, \mu_2^{(1)}]. \quad (3.8)$$

Here we introduce a degree even product

$$\mu_2^{(1)} \equiv \frac{1}{3} \left(\xi M_2^{(0)} - M_2^{(0)} (\xi \otimes \mathbb{I} + \mathbb{I} \otimes \xi) \right), \quad (3.9)$$

with $\xi \equiv \oint \frac{dz}{2\pi i} f(z) \xi(z)$, which also satisfies

$$\mathbf{M}_2^{(0)} = [\mathbf{\eta}, \mu_2^{(1)}], \quad (3.10)$$

where $\mathbf{\eta}$ is the coderivation derived from the η zero mode. The fact that $M_2^{(1)}$ is BRST exact means that it can be generated by a gauge transformation through the space of A_∞ structures [18]. So to find a solution to the A_∞ relations, all we have to do is complete the construction of the gauge transformation so as to ensure that $M_3^{(2)}, M_4^{(3)}, \dots$ are in the small Hilbert space. The gauge transformation is defined by $\mu_2^{(1)}$ and an array of higher-point products $\mu_l^{(k)}$ of even degree. We will call these “gauge products.”¹²

The first nonlinear correction to the gauge transformation determines the 3-product $M_3^{(2)}$, via the formula

$$\mathbf{M}_3^{(2)} = \frac{1}{2} \left([\mathbf{Q}, \mu_3^{(2)}] + [\mathbf{M}_2^{(1)}, \mu_2^{(1)}] \right), \quad (3.11)$$

where we introduce a gauge 3-product $\mu_3^{(2)}$ with picture number two. Plugging in and using the Jacobi identity, it is easy to see that the 3rd A_∞ relation is identically satisfied:

$$0 = \frac{1}{2} [\mathbf{M}_2^{(1)}, \mathbf{M}_2^{(1)}] + [\mathbf{Q}, \mathbf{M}_3^{(2)}]. \quad (3.12)$$

¹¹Note that the cohomology of Q and η is trivial in the large Hilbert space.

¹²The notation and terminology for products used here differs from [18]. The relation between here and there is $M_{n+1}^{(n)} = M_n$, $\mu_{n+2}^{(n+1)} = \overline{M}_{n+2}$ and $M_{n+2}^{(n)} = m_{n+2}$.

However, the term $[\mathbf{Q}, \boldsymbol{\mu}_3^{(2)}]$ in (3.11) does not play a role for this purpose. This term is needed for a different reason: to ensure that $M_3^{(2)}$ lives in the small Hilbert space. Let's define a degree odd 3-product $M_3^{(1)}$ with picture 1, satisfying

$$\mathbf{M}_3^{(1)} = [\boldsymbol{\eta}, \boldsymbol{\mu}_3^{(2)}]. \quad (3.13)$$

Requiring $M_3^{(2)}$ to be in the small Hilbert space implies

$$\begin{aligned} [\boldsymbol{\eta}, \mathbf{M}_3^{(2)}] &= 0 = \frac{1}{2} \left(-[\mathbf{Q}, \mathbf{M}_3^{(1)}] - [\mathbf{M}_2^{(1)}, \mathbf{M}_2^{(0)}] \right), \\ &= \frac{1}{2} \left[\mathbf{Q}, -\mathbf{M}_3^{(1)} + [\mathbf{M}_2^{(0)}, \boldsymbol{\mu}_2^{(1)}] \right]. \end{aligned} \quad (3.14)$$

Therefore $M_3^{(1)}$ must satisfy

$$\mathbf{M}_3^{(1)} = [\mathbf{Q}, \boldsymbol{\mu}_3^{(1)}] + [\mathbf{M}_2^{(0)}, \boldsymbol{\mu}_2^{(1)}], \quad (3.15)$$

where we introduce yet another gauge 3-product $\mu_3^{(1)}$ with picture number 1. In [18] it was consistent to set $\mu_3^{(1)} = 0$ because Witten's open string star product is associative. Now we will not assume that $M_2^{(0)}$ is associative, so the term $[\mathbf{Q}, \boldsymbol{\mu}_3^{(1)}]$ is needed to make sure that $M_3^{(1)}$ is in the small Hilbert space, as is required by (3.13). We define $\mu_3^{(1)}$ by the relation

$$2\mathbf{M}_3^{(0)} = [\boldsymbol{\eta}, \boldsymbol{\mu}_3^{(1)}], \quad (3.16)$$

where $\mathbf{M}_3^{(0)}$ is the bosonic 3-product. Then taking η of (3.14) implies

$$0 = [\mathbf{Q}, \mathbf{M}_3^{(0)}] + \frac{1}{2} [\mathbf{M}_2^{(0)}, \mathbf{M}_2^{(0)}]. \quad (3.17)$$

This is nothing but the 3rd A_∞ relation for the bosonic products. The upshot is that we can determine $M_3^{(2)}$ for Witten's superstring field theory by climbing a "ladder" of products and gauge products starting from $M_3^{(0)}$ as follows:

$$\mathbf{M}_3^{(0)} = \text{given}, \quad (3.18)$$

$$\mu_3^{(1)} = \frac{1}{2} \left(\xi M_3^{(0)} - M_3^{(0)} (\xi \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \xi \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \xi) \right), \quad (3.19)$$

$$\mathbf{M}_3^{(1)} = [\mathbf{Q}, \boldsymbol{\mu}_3^{(1)}] + [\mathbf{M}_2^{(0)}, \boldsymbol{\mu}_2^{(1)}], \quad (3.20)$$

$$\mu_3^{(2)} = \frac{1}{4} \left(\xi M_3^{(1)} - M_3^{(1)} (\xi \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \xi \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \xi) \right), \quad (3.21)$$

$$\mathbf{M}_3^{(2)} = \frac{1}{2} \left([\mathbf{Q}, \boldsymbol{\mu}_3^{(2)}] + [\mathbf{M}_2^{(1)}, \boldsymbol{\mu}_2^{(1)}] \right). \quad (3.22)$$

The second and fourth equations invert (3.16) and (3.13) by placing a ξ insertion once on each output of the respective 3-product. Incidentally, we construct $M_2^{(1)}$ by climbing a similar ladder

$$\mathbf{M}_2^{(0)} = \text{given}, \quad (3.23)$$

$$\mu_2^{(1)} = \frac{1}{3} \left(\xi M_2^{(0)} - M_2^{(0)} (\xi \otimes \mathbb{I} + \mathbb{I} \otimes \xi) \right), \quad (3.24)$$

$$\mathbf{M}_2^{(1)} = [\mathbf{Q}, \mu_2^{(1)}], \quad (3.25)$$

but in this case it was easier to postulate the final answer from the beginning, (3.4).

Proceeding in this way, it is not difficult to anticipate that the $(n+1)$ -string product $M_{n+1}^{(n)}$ of Witten's superstring field theory can be constructed by ascending a ladder of $n+1$ products

$$M_{n+1}^{(0)}, M_{n+1}^{(1)}, \dots, M_{n+1}^{(n)}, \quad (3.26)$$

interspersed with n gauge products

$$\mu_{n+1}^{(1)}, \mu_{n+1}^{(2)}, \dots, \mu_{n+1}^{(n)}, \quad (3.27)$$

adding picture number one step at a time. Thus we will have a recursive solution to the A_∞ relations, expressed in terms of a “triangle” of products, as shown in figure 3.1.

3.2 All Vertices

We now explain how to determine the vertices to all orders. We start by collecting superstring products into a generating function

$$\mathbf{M}^{[0]}(t) \equiv \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+1}^{(n)}, \quad (3.28)$$

so that the $(n+1)$ st superstring product can be extracted by looking at the coefficient of t^n . Here we place an upper index on the generating function (in square brackets) to indicate the “deficit” in picture number of the products relative to what is needed for the superstring. In this case, of course, the deficit is zero. The superstring products must satisfy two properties. First, they must be in the small Hilbert space, and second, they must satisfy the A_∞ relations:

$$[\mathfrak{n}, \mathbf{M}^{[0]}(t)] = 0, \quad [\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)] = 0. \quad (3.29)$$

Expanding the second equation in powers of t gives the A_∞ relations as written in (2.14). To solve the A_∞ relations, we postulate the differential equation

$$\frac{\partial}{\partial t} \mathbf{M}^{[0]}(t) = [\mathbf{M}^{[0]}(t), \mu^{[0]}(t)], \quad (3.30)$$

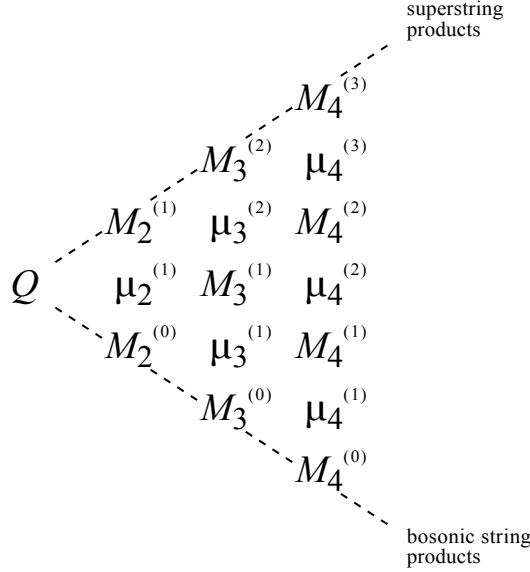


Figure 3.1: The products $M_n^{(n-1)}$ of Witten's superstring field theory are derived from the products $M_n^{(0)}$ of open bosonic string field theory by constructing a triangular array of products of intermediate picture number.

where

$$\mu^{[0]}(t) = \sum_{n=0}^{\infty} t^n \mu_{n+2}^{(n+1)} \quad (3.31)$$

is a generating function for “deficit-free” gauge products. Expanding (3.30) in powers of t gives previous formulas (3.8) and (3.13) for the 2-product and the 3-product. Note that this differential equation implies

$$\frac{\partial}{\partial t} [\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)] = [[\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)], \mu^{[0]}(t)]. \quad (3.32)$$

Since this is homogeneous in $[\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)]$, the A_{∞} relations follow immediately from the fact that $[\mathbf{M}^{[0]}(t), \mathbf{M}^{[0]}(t)] = 0$ holds at $t = 0$ (since \mathbf{Q} is nilpotent). Note that the generating function (3.28) can also be interpreted as defining a 1-parameter family of A_{∞} algebras, where the parameter t is the open string coupling constant [18]. In this context, the differential equation (3.30) says that changes of the coupling constant are implemented by a gauge transformation through the space of A_{∞} structures, and $\mu^{[0]}(t)$ is the infinitesimal gauge parameter.

The statement that the coupling constant is “pure gauge” normally means that the cubic and higher order vertices can be removed by field redefinition, and the scattering amplitudes vanish [31]. This does not happen here because $\mu^{[0]}(t)$ is in the large Hilbert space, and therefore does not define an “admissible” gauge parameter. But then the

nontrivial condition is that the superstring products are in the small Hilbert space despite the fact that the gauge transformation defining them is not. To see what this condition implies, take $\boldsymbol{\eta}$ of the differential equation (3.30) to find

$$[\mathbf{M}^{[0]}(t), \mathbf{M}^{[1]}(t)] = 0, \quad (3.33)$$

where

$$\mathbf{M}^{[1]}(t) = [\boldsymbol{\eta}, \boldsymbol{\mu}^{[0]}(t)] = \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+2}^{(n)} \quad (3.34)$$

is the generating function for products with a single picture deficit. Now we can solve (3.33) by postulating a new differential equation

$$\frac{\partial}{\partial t} \mathbf{M}^{[1]}(t) = [\mathbf{M}^{[0]}(t), \boldsymbol{\mu}^{[1]}(t)] + [\mathbf{M}^{[1]}(t), \boldsymbol{\mu}^{[0]}(t)], \quad (3.35)$$

where

$$\boldsymbol{\mu}^{[1]}(t) = \sum_{n=0}^{\infty} t^n \boldsymbol{\mu}_{n+3}^{(n+1)} \quad (3.36)$$

is a generating function for gauge products with a single picture deficit. Now we are beginning to see the outlines of a recursion. Taking $\boldsymbol{\eta}$ of (3.35) implies a constraint on the generating function for products with two picture deficits $\mathbf{M}^{[2]}(t)$, which can be solved by postulating yet another differential equation, and so on. The full recursion is most compactly expressed by packaging the generating functions $\mathbf{M}^{[m]}(t)$ and $\boldsymbol{\mu}^{[m]}(t)$ together in a power series in a new parameter s :

$$\mathbf{M}(s, t) \equiv \sum_{m=0}^{\infty} s^m \mathbf{M}^{[m]}(t) = \sum_{m,n=0}^{\infty} s^m t^n \mathbf{M}_{m+n+1}^{(n)}, \quad (3.37)$$

$$\boldsymbol{\mu}(s, t) \equiv \sum_{m=0}^{\infty} s^m \boldsymbol{\mu}^{[m]}(t) = \sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{\mu}_{m+n+2}^{(n+1)}. \quad (3.38)$$

Note that powers of t count the picture number, and powers of s count the deficit in picture number. At $t = 0$ $\mathbf{M}(s, t)$ reduces to a generating function for products of the bosonic string, and at $s = 0$ it reduces to a generating function for products of the superstring:

$$\mathbf{M}(s, 0) = \sum_{n=0}^{\infty} s^n \mathbf{M}_{n+1}^{(0)}, \quad (3.39)$$

$$\mathbf{M}(0, t) = \mathbf{M}^{[0]}(t) = \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+1}^{(n)}. \quad (3.40)$$

The recursion then emerges from expansion of a *pair* of differential equations

$$\frac{\partial}{\partial t} \mathbf{M}(s, t) = [\mathbf{M}(s, t), \boldsymbol{\mu}(s, t)], \quad (3.41)$$

$$\frac{\partial}{\partial s} \mathbf{M}(s, t) = [\boldsymbol{\eta}, \boldsymbol{\mu}(s, t)]. \quad (3.42)$$

Note that these equations imply

$$\frac{\partial}{\partial t} [\mathbf{M}(s, t), \mathbf{M}(s, t)] = [[\mathbf{M}(s, t), \mathbf{M}(s, t)], \boldsymbol{\mu}(s, t)], \quad (3.43)$$

$$\frac{\partial}{\partial t} [\boldsymbol{\eta}, \mathbf{M}(s, t)] = [[\boldsymbol{\eta}, \mathbf{M}(s, t)], \boldsymbol{\mu}(s, t)] - \frac{1}{2} \frac{\partial}{\partial s} [\mathbf{M}(s, t), \mathbf{M}(s, t)]. \quad (3.44)$$

Since the first equation is homogeneous in $[\mathbf{M}(s, t), \mathbf{M}(s, t)]$, the A_∞ relations for the bosonic products at $t = 0$ implies $[\mathbf{M}(s, t), \mathbf{M}(s, t)] = 0$ for all s and t . Thus the second equation (3.44) becomes homogeneous in $[\boldsymbol{\eta}, \mathbf{M}(s, t)]$, and the fact that the bosonic products are in the small Hilbert space at $t = 0$ implies that all products are in the small Hilbert space. Thus

$$[\mathbf{M}(s, t), \mathbf{M}(s, t)] = 0, \quad [\boldsymbol{\eta}, \mathbf{M}(s, t)] = 0. \quad (3.45)$$

Setting $s = 0$ we recover (3.29). Therefore, solving (3.41) and (3.42) automatically determines a set of superstring products which live in the small Hilbert space and satisfy the A_∞ relations.

Now all we need to do is solve the differential equations (3.41) and (3.42) to determine the products. Expanding (3.41) in s, t and reading off the coefficient of $s^m t^n$ gives the formula:

$$\mathbf{M}_{m+n+2}^{(n+1)} = \frac{1}{n+1} \sum_{k=0}^n \sum_{l=0}^m [\mathbf{M}_{k+l+1}^{(k)}, \boldsymbol{\mu}_{m+n-k-l+2}^{(n-k+1)}]. \quad (3.46)$$

This determines the product $M_{m+n+2}^{(n+1)}$ if we are given gauge products

$$\mu_l^{(k)}, \quad 1 \leq k \leq n+1, \quad k+1 \leq l \leq k+m+1, \quad (3.47)$$

and the lower order products

$$M_l^{(k)}, \quad 0 \leq k \leq n, \quad k+1 \leq l \leq k+m+1. \quad (3.48)$$

The lower order products are either again determined by (3.46), or they are products of the bosonic string, which we assume are given. So now we must find the gauge products $\mu_l^{(k)}$. Expanding (3.42) gives

$$[\boldsymbol{\eta}, \boldsymbol{\mu}_{m+n+2}^{(n+1)}] = (m+1) \mathbf{M}_{m+n+2}^{(n)}. \quad (3.49)$$

This equation will determine $\mu_{m+n+2}^{(n+1)}$ in terms of $M_{m+n+2}^{(m)}$. The solution is not unique. However there is a natural ansatz preserving cyclicity:

$$\mu_{m+n+2}^{(m+1)} = \frac{n+1}{m+n+3} \left(\xi M_{m+n+2}^{(m)} - M_{m+n+2}^{(m)} \sum_{k=0}^{m+n+1} \mathbb{I}^{\otimes k} \otimes \xi \otimes \mathbb{I}^{\otimes m+n+1-k} \right). \quad (3.50)$$

or, more compactly, we can write $\mu_{m+n+2}^{(m+1)} = (n+1)\xi \circ \mathbf{M}_{m+n+2}^{(m)}$ where $\xi \circ$ denotes the operation of taking the average of ξ acting on the output and on each input of the product. This ansatz works assuming $M_{m+n+2}^{(m)}$ is in the small Hilbert space, but we have to show that the ansatz is consistent with that assumption. To this end, note that if (3.41) is satisfied and the gauge products are defined by (3.50), we have the relation

$$\frac{\partial}{\partial t}[\mathfrak{n}, \mathbf{M}(s, t)] = [[\mathfrak{n}, \mathbf{M}(s, t)], \mu(s, t)] + \left[\mathbf{M}(s, t), \frac{\partial}{\partial s} \xi \circ [\mathfrak{n}, \mathbf{M}(s, t)] \right]. \quad (3.51)$$

Since this equation is homogeneous in $[\mathfrak{n}, \mathbf{M}(s, t)]$, (3.50) implies that all products must be in the small Hilbert space.

The construction is recursive. Assume that we have already constructed all products $M_m^{(k)}$ and gauge products $\mu_m^{(k)}$ with $m \leq n$ inputs and with all picture numbers. Then we construct the $(n+1)$ st product of Witten's superstring field theory by climbing a ladder of products and gauge products, defined by equations (3.46) and (3.49):

$$\begin{aligned} \mathbf{M}_{n+1}^{(0)} &= \text{given}, \\ \mu_{n+1}^{(1)} &= \frac{n}{n+2} \left(\xi M_{n+1}^{(0)} - M_{n+1}^{(0)} \sum_{k=0}^n \mathbb{I}^{\otimes k} \otimes \xi \otimes \mathbb{I}^{\otimes n-k} \right), \\ \mathbf{M}_{n+1}^{(1)} &= [\mathbf{Q}, \mu_{n+1}^{(1)}] + [\mathbf{M}_2^{(0)}, \mu_n^{(1)}] + \dots + [\mathbf{M}_n^{(0)}, \mu_2^{(1)}], \\ \mu_{n+1}^{(2)} &= \frac{n-1}{n+2} \left(\xi M_{n+1}^{(1)} - M_{n+1}^{(1)} \sum_{k=0}^n \mathbb{I}^{\otimes k} \otimes \xi \otimes \mathbb{I}^{\otimes n-k} \right), \\ \mathbf{M}_{n+1}^{(2)} &= \frac{1}{2} \left([\mathbf{Q}, \mu_{n+1}^{(2)}] + [\mathbf{M}_2^{(0)}, \mu_n^{(2)}] + [\mathbf{M}_2^{(1)}, \mu_n^{(1)}] + \dots \right. \\ &\quad \left. + [\mathbf{M}_{n-1}^{(0)}, \mu_3^{(2)}] + [\mathbf{M}_{n-1}^{(1)}, \mu_3^{(1)}] + [\mathbf{M}_n^{(1)}, \mu_2^{(1)}] \right), \\ &\vdots \\ \mu_{n+1}^{(n)} &= \frac{1}{n+2} \left(\xi M_{n+1}^{(n)} - M_{n+1}^{(n)} \sum_{k=0}^n \mathbb{I}^{\otimes k} \otimes \xi \otimes \mathbb{I}^{\otimes n-k} \right), \\ \mathbf{M}_{n+1}^{(n)} &= \frac{1}{n} \left([\mathbf{Q}, \mu_{n+1}^{(n)}] + [\mathbf{M}_2^{(1)}, \mu_n^{(n-1)}] + \dots + [\mathbf{M}_n^{(n-1)}, \mu_2^{(1)}] \right). \end{aligned} \quad (3.52)$$

The final step in this ladder is the $n + 1$ -string product of Witten's open superstring field theory. Incidentally, note that the nature of this construction guarantees that the superstring products will define cyclic vertices if the bosonic products do (see appendix B of [18]).

4 NS Heterotic String

Our analysis of the open superstring almost immediately generalizes to a construction of heterotic string field theory in the NS sector. An alternative formulation of this theory, using the large Hilbert space, is described in [7, 8]. The closed string field is a degree (and Grassmann) even NS state Φ in the superconformal field theory of a heterotic string. Note that the $\beta\gamma$ ghosts and picture only reside in the leftmoving sector. The string field has ghost number 2 and picture number -1 , and satisfies the b_0^- and level matching constraints (2.47). An on-shell state in Siegel gauge takes the form

$$\Phi \sim c\bar{c}e^{-\phi}\mathcal{O}^m(0,0), \quad (4.1)$$

where \mathcal{O}^m is a matter primary operator with left/rightmoving dimension $(\frac{1}{2}, 1)$. The symplectic form (2.45) is nonvanishing only on states whose ghost number adds up to five and whose picture number adds up to -2 .

The action is defined by a sequence of degree odd closed string products

$$L_1^{(0)} = Q, \quad L_2^{(1)}, \quad L_3^{(2)}, \quad L_4^{(3)}, \quad \dots, \quad (4.2)$$

satisfying the relations of a cyclic L_∞ algebra. Just like in the open string, the n th closed string product must have picture $n - 1$ to define a nonvanishing vertex. We construct the products by placing picture changing operators on the products of the closed bosonic string

$$L_1^{(0)} = Q, \quad L_2^{(0)}, \quad L_3^{(0)}, \quad L_4^{(0)}, \quad \dots, \quad (4.3)$$

which, of course, have vanishing picture. The explicit definition of the closed bosonic string products is an intricate story [2, 32, 33, 34, 35], but for our purposes all we need to know is: 1) they satisfy the relations of a cyclic L_∞ algebra, 2) they are in the small Hilbert space, 3) they carry vanishing picture number, and 4) they are consistent with the b_0^- and L_0^- constraints.

The problem we need to solve appears completely analogous to the open superstring. Aside from replacing tensor products with wedge products, there is one minor difference. Since the products of the heterotic string must respect the b_0^- and L_0^- constraints, the picture changing operator X in the 2-product

$$L_2^{(1)}(\Phi_1, \Phi_2) = \frac{1}{3} \left(X L_2^{(0)}(\Phi_1, \Phi_2) + L_2^{(0)}(X\Phi_1, \Phi_2) + L_2^{(0)}(\Phi_1, X\Phi_2) \right) \quad (4.4)$$

must be identified with the zero mode X_0 . This way, we can pull b_0^- and L_0^- past X_0 to act on $L_2^{(0,0)}$, which vanishes. More generally, we must construct closed superstring products using the ξ zero mode

$$\xi = \xi_0 = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z} \xi(z), \quad (4.5)$$

rather than a more general charge which would be consistent for the open string.

Following the discussion of the open superstring, we introduce a “triangle” of products

$$L_{n+1}^{(k)}, \quad 0 \leq n \leq \infty, \quad 0 \leq k \leq n, \quad (4.6)$$

and gauge products,

$$\lambda_{n+2}^{(k+1)}, \quad 0 \leq n \leq \infty, \quad 0 \leq k \leq n \quad (4.7)$$

of intermediate picture indicated in the upper index. We build the $(n+1)$ -heterotic string product $L_{n+1}^{(n)}$ by climbing a “ladder” of products

$$L_{n+1}^{(0)}, \lambda_{n+1}^{(1)}, L_{n+1}^{(1)}, \dots, \lambda_{n+1}^{(n)}, L_{n+1}^{(n)}, \quad (4.8)$$

adding picture one step at a time. Each step is prescribed by the closed string analogues of equations (3.46) and (3.49):

$$\mathbf{L}_{m+n+2}^{(m+1)} = \frac{1}{m+1} \sum_{k=0}^m \sum_{l=0}^n [\mathbf{L}_{k+l+1}^{(k)}, \lambda_{m+n-k-l+2}^{(m-k+1)}] \quad (4.9)$$

$$\lambda_{m+n+2}^{(m+1)} = \frac{n+1}{m+n+3} \left(\xi_0 L_{m+n+2}^{(m)} - L_{m+n+2}^{(m)} (\xi_0 \wedge \mathbb{I}_{m+n+1}) \right). \quad (4.10)$$

The only differences from the open superstring are that the coderivations act on the symmetrized tensor algebra, and ξ has been replaced by ξ_0 .

5 NS-NS Closed Superstring

We are now ready to discuss the NS-NS sector of Type II closed superstring field theory. A recent proposal for defining this theory in the large Hilbert space appears in [21]. The closed string field is a degree even (and Grassmann even) NS-NS state Φ in the superconformal field theory of a type II superstring. Now $\beta\gamma$ ghosts and picture occupy both the leftmoving and rightmoving sectors. The string field has ghost number 2, satisfies the b_0^- and L_0^- constraints (2.47), and has left/rightmoving picture number $(-1, -1)$. On-shell states in Siegel gauge take the form

$$\Phi \sim c\bar{c}e^{-\phi}e^{-\bar{\phi}}\mathcal{O}^m(0,0), \quad (5.1)$$

where \mathcal{O}^m is a superconformal matter primary of weight $(\frac{1}{2}, \frac{1}{2})$. The symplectic form (2.45) is nonvanishing on states of ghost number 5 and left/right picture $(-2, -2)$.

The theory is defined by a sequence of degree odd closed string products

$$L_1^{(0,0)} = Q, \quad L_2^{(1,1)}, \quad L_3^{(2,2)}, \quad L_4^{(3,3)}, \quad \dots, \quad (5.2)$$

satisfying the relations of a cyclic L_∞ algebra. The $(n+1)$ st closed string product must have left/right picture (n, n) . These products should be constructed from the products of the closed bosonic string,

$$L_1^{(0,0)} = Q, \quad L_2^{(0,0)}, \quad L_3^{(0,0)}, \quad L_4^{(0,0)}, \quad \dots, \quad (5.3)$$

which have vanishing picture. Note that we add an extra index to indicate rightmoving picture. Now the situation is somewhat different from the open string, since we need to add twice as much picture and we need to pay attention to how it is distributed between leftmoving and rightmoving sectors. However, it is not difficult to guess what the 2-product should look like. Starting with $L_2^{(0,0)}$, we surround it once with a leftmoving picture changing operator X_0 , and again a rightmoving picture changing operator \overline{X}_0 , to produce the expression

$$\begin{aligned} L_2^{(1,1)}(\Phi_1, \Phi_2) = & \frac{1}{9} \left(X_0 \overline{X}_0 L_2^{(0,0)}(\Phi_1, \Phi_2) + X_0 L_2^{(0,0)}(\overline{X}_0 \Phi_1, \Phi_2) + X_0 L_2^{(0,0)}(\Phi_1, \overline{X}_0 \Phi_2) \right. \\ & + \overline{X}_0 L_2^{(0,0)}(X_0 \Phi_1, \Phi_2) + L_2^{(0,0)}(X_0 \overline{X}_0 \Phi_1, \Phi_2) + L_2^{(0,0)}(X_0 \Phi_1, \overline{X}_0 \Phi_2) \\ & \left. + \overline{X}_0 L_2^{(0,0)}(\Phi_1, X_0 \Phi_2) + L_2^{(0,0)}(\overline{X}_0 \Phi_1, X_0 \Phi_2) + L_2^{(0,0)}(\Phi_1, X_0 \overline{X}_0 \Phi_2) \right). \end{aligned} \quad (5.4)$$

Note that since X_0 and \overline{X}_0 commute it does not matter which order we apply them to the bosonic product.

5.1 Asymmetric Construction

The easiest solution for the closed superstring is to apply the open string construction twice: The first time to get the correct picture number for leftmovers and a second time to get the correct picture number for the rightmovers. More specifically we proceed as follows. Starting with the bosonic product $L_{n+1}^{(0,0)}$ we climb a “ladder” of products and gauge products

$$L_{n+1}^{(0,0)}, \quad \lambda_{n+1}^{(1,0)}, \quad L_{n+1}^{(1,0)}, \quad \dots, \quad \lambda_{n+1}^{(n,0)}, \quad L_{n+1}^{(n,0)}, \quad (5.5)$$

using (4.9) and (4.10) (with an extra spectator index for rightmoving picture). At the top of the ladder, the product $L_{n+1}^{(n,0)}$ has the required leftmoving picture, but the rightmoving

picture is still absent. So we take $L_{n+1}^{(n,0)}$ as the input for a second set of recursions which add rightmoving picture. Starting with $L_{n+1}^{(n,0)}$ we climb a second “ladder”

$$L_{n+1}^{(n,0)}, \lambda_{n+1}^{(n,1)}, L_{n+1}^{(n,1)}, \dots, \lambda_{n+1}^{(n,n)}, L_{n+1}^{(n,n)}, \quad (5.6)$$

again using (4.9) and (4.10), but this time the leftmoving picture is a spectator index, and the right moving zero mode $\bar{\xi}_0$ appears in (4.10) rather than the leftmoving one. Thus, for example the 2-product is constructed by climbing two ladders:

$$\begin{aligned} \text{first ladder} & \left\{ \begin{array}{l} \mathbf{L}_2^{(0,0)} = \text{given}, \\ \lambda_2^{(1,0)} = \frac{1}{3} \left(\xi_0 L_2^{(0,0)} - L_2^{(0,0)} (\xi_0 \wedge \mathbb{I}) \right), \\ \mathbf{L}_2^{(1,0)} = [\mathbf{Q}, \lambda_2^{(1,0)}], \end{array} \right. \\ \text{second ladder} & \left\{ \begin{array}{l} \mathbf{L}_2^{(1,0)} = \text{given by first ladder}, \\ \lambda_2^{(1,1)} = \frac{1}{3} \left(\bar{\xi}_0 L_2^{(1,0)} - L_2^{(1,0)} (\bar{\xi}_0 \wedge \mathbb{I}) \right), \\ \mathbf{L}_2^{(1,1)} = [\mathbf{Q}, \lambda_2^{(1,1)}]. \end{array} \right. \end{aligned} \quad (5.7)$$

This is the simplest construction we have found the NS-NS superstring, in the sense that it requires the fewest auxiliary products of intermediate picture number in defining the recursion. However, it suffers from a curious asymmetry between left and rightmoving picture changing operators. This asymmetry first appears in $L_3^{(2,2)}$, which for example has a term of the form

$$\bar{X}_0^2 L_2^{(0,0)} \left(X_0 \xi_0 L_2^{(0,0)} (\Phi_1, \Phi_2), \Phi_3 \right), \quad (5.8)$$

and no corresponding term with left and rightmovers reversed.

5.2 Symmetric Construction

To restore symmetry between left and rightmovers we consider a different solution of the L_∞ relations. To motivate the structure, consider the 2-product $L_2^{(1,1)}$ written in the form

$$\mathbf{L}_2^{(1,1)} = \frac{1}{2} [\mathbf{Q}, \lambda_2^{(1,1)} + \bar{\lambda}_2^{(1,1)}]. \quad (5.9)$$

Now we have introduced *two* gauge products. The first $\lambda_2^{(1,1)}$ will be called a “left” gauge product, and is defined by replacing X_0 in the expression (5.4) for $L_2^{(1,1)}$ with ξ_0 . The second $\bar{\lambda}_2^{(1,1)}$ will be called a “right” gauge product, and is defined by replacing \bar{X}_0 in $L_2^{(1,1)}$ with $\bar{\xi}_0$. Once we act with \mathbf{Q} , $\lambda_2^{(1,1)}$ and $\bar{\lambda}_2^{(1,1)}$ produce the same expression (hence

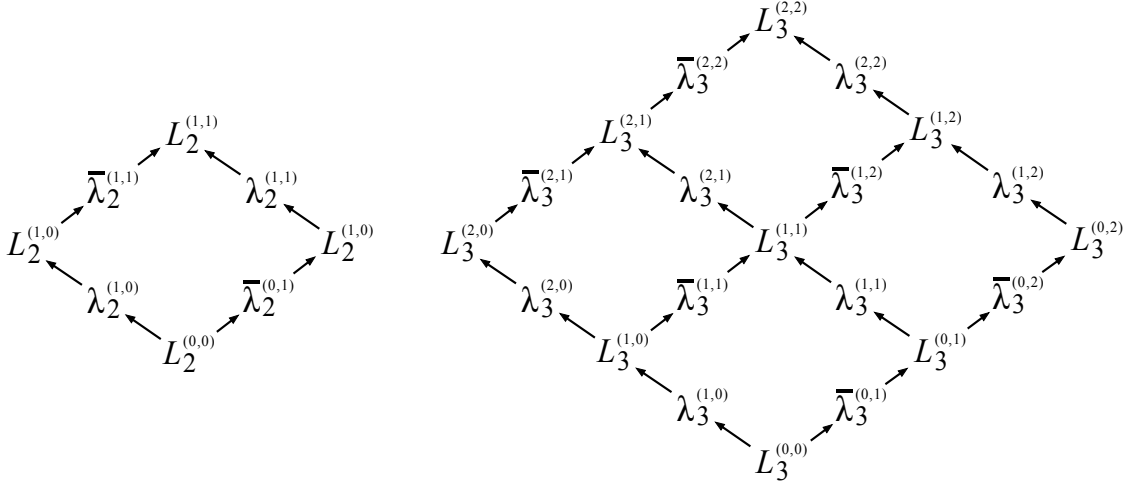


Figure 5.1: “Diamonds” of products and gauge products needed to construct the 2-product and 3-product of NS-NS closed superstring field theory.

the factor of $1/2$), but the advantage of this decomposition is that left/right symmetry is manifest. Denoting the left/rightmoving eta zero modes by η and $\bar{\eta}$, we have the relations

$$[\eta, \lambda_2^{(1,1)}] = \mathbf{L}_2^{(0,1)} \quad [\bar{\eta}, \bar{\lambda}_2^{(1,1)}] = \mathbf{L}_2^{(1,0)} \quad (5.10)$$

$$[\eta, \bar{\lambda}_2^{(1,1)}] = 0 \quad [\bar{\eta}, \lambda_2^{(1,1)}] = 0. \quad (5.11)$$

Note that the left gauge product $\lambda_2^{(1,1)}$ is in the “rightmoving small Hilbert space,” while the right gauge product $\bar{\lambda}_2^{(1,1)}$ is in the “leftmoving small Hilbert space.” The products $L_2^{(1,0)}$ and $L_2^{(0,1)}$ now carry a single X_0 or \bar{X}_0 insertion, respectively. Pulling \mathbf{Q} out we can write

$$\mathbf{L}_2^{(1,0)} = [\mathbf{Q}, \lambda_2^{(1,0)}], \quad \mathbf{L}_2^{(0,1)} = [\mathbf{Q}, \bar{\lambda}_2^{(0,1)}], \quad (5.12)$$

where $\lambda_2^{(1,0)}$ and $\bar{\lambda}_2^{(0,1)}$ are left/right gauge products satisfying

$$[\eta, \lambda_2^{(1,0)}] = [\bar{\eta}, \bar{\lambda}_2^{(0,1)}] = \mathbf{L}_2^{(0,0)} \quad (5.13)$$

$$[\bar{\eta}, \lambda_2^{(1,0)}] = [\eta, \bar{\lambda}_2^{(0,1)}] = 0, \quad (5.14)$$

and $L_2^{(0,0)}$ is the product of the bosonic string. In this way the superstring product $L_2^{(1,1)}$ is derived by filling a “diamond” of products and gauge products, as shown in figure 5.1.

Also shown is a “diamond” illustrating the derivation of the 3-product, which has four “cells” giving a total of 21 intermediate products. The explicit formulas associated with this diagram are difficult to guess, so we will proceed to motivate the general construction.

To find the closed superstring product $L_{n+1}^{(n,n)}$, we need a diamond consisting of $(n+1)^2$ products

$$L_{n+1}^{(p,q)}, \quad 0 \leq p, q \leq n, \quad (5.15)$$

$n(n+1)$ left gauge products

$$\lambda_{n+1}^{(p,q)}, \quad \begin{aligned} 1 \leq p \leq n, \\ 0 \leq q \leq n, \end{aligned} \quad (5.16)$$

and $n(n+1)$ right gauge products

$$\bar{\lambda}_{n+1}^{(p,q)}, \quad \begin{aligned} 0 \leq p \leq n, \\ 1 \leq q \leq n. \end{aligned} \quad (5.17)$$

We would like to package the products into three generating functions

$$\mathbf{L}(s, \bar{s}, t), \quad \boldsymbol{\lambda}(s, \bar{s}, t), \quad \bar{\boldsymbol{\lambda}}(s, \bar{s}, t), \quad (5.18)$$

which depend on three variables, corresponding to the three indices characterizing the products. The variable t counts the total picture number, s the deficit in leftmoving picture number, and \bar{s} the deficit in rightmoving picture number. Thus we have

$$\mathbf{L}(s, \bar{s}, t) = \sum_{N=0}^{\infty} \sum_{i,j=0}^N t^{i+j} s^{N-i} \bar{s}^{N-j} \mathbf{L}_{N+1}^{(i,j)}, \quad (5.19)$$

$$\boldsymbol{\lambda}(s, \bar{s}, t) = \sum_{N=0}^{\infty} \sum_{i=0}^N \sum_{j=0}^{N+1} t^{i+j} s^{N-i} \bar{s}^{N+1-j} \boldsymbol{\lambda}_{N+2}^{(i+1,j)}, \quad (5.20)$$

$$\bar{\boldsymbol{\lambda}}(s, \bar{s}, t) = \sum_{N=0}^{\infty} \sum_{i=0}^{N+1} \sum_{j=0}^N t^{i+j} s^{N+1-i} \bar{s}^{N-j} \bar{\boldsymbol{\lambda}}_{N+2}^{(i,j+1)}. \quad (5.21)$$

The ranges of summation here are complicated in comparison to what appears in the generating functions of the open string. The reason is that the closed superstring has left/rightmoving sectors with separate picture numbers, but not separate notions of multiplication. However, we can simplify these formulas by formally introducing an extra index to indicate “rightmoving multiplication:”

$$\mathbf{L}_{m+1,n+1}^{(p,q)} \equiv \delta_{m,n} \mathbf{L}_{m+1}^{(p,q)}, \quad (5.22)$$

$$\boldsymbol{\lambda}_{m+2,n+1}^{(p,q)} \equiv \delta_{m+1,n} \boldsymbol{\lambda}_{m+2}^{(p,q)}, \quad (5.23)$$

$$\bar{\boldsymbol{\lambda}}_{m+1,n+2}^{(p,q)} \equiv \delta_{m,n+1} \bar{\boldsymbol{\lambda}}_{n+2}^{(p,q)}, \quad (5.24)$$

with a Kronecker delta to identify multiplication between the left and right. Then the generating functions take the form:

$$\mathbf{L}(t, s, \bar{s}) = \sum_{m,n=0}^{\infty} \sum_{p,q=0}^{\infty} \left(t^m s^n \right) \left(t^p \bar{s}^q \right) \mathbf{L}_{m+n+1,p+q+1}^{(m,p)}, \quad (5.25)$$

$$\boldsymbol{\lambda}(t, s, \bar{s}) = \sum_{m,n=0}^{\infty} \sum_{p,q=0}^{\infty} \left(t^m s^n \right) \left(t^p \bar{s}^q \right) \boldsymbol{\lambda}_{m+n+2,p+q+1}^{(m+1,p)}, \quad (5.26)$$

$$\bar{\boldsymbol{\lambda}}(t, s, \bar{s}) = \sum_{m,n=0}^{\infty} \sum_{p,q=0}^{\infty} \left(t^m s^n \right) \left(t^p \bar{s}^q \right) \bar{\boldsymbol{\lambda}}_{m+n+1,p+q+2}^{(m,p+1)}. \quad (5.27)$$

The solution to the L_{∞} relations is defined by the system of equations

$$\frac{\partial}{\partial t} \mathbf{L}(s, \bar{s}, t) = \left[\mathbf{L}(s, \bar{s}, t), \boldsymbol{\lambda}(s, \bar{s}, t) + \bar{\boldsymbol{\lambda}}(s, \bar{s}, t) \right], \quad (5.28)$$

$$\frac{\partial}{\partial s} \mathbf{L}(s, \bar{s}, t) = [\boldsymbol{\eta}, \boldsymbol{\lambda}(s, \bar{s}, t)], \quad [\bar{\boldsymbol{\eta}}, \boldsymbol{\lambda}(s, \bar{s}, t)] = 0, \quad (5.29)$$

$$\frac{\partial}{\partial \bar{s}} \mathbf{L}(s, \bar{s}, t) = [\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\lambda}}(s, \bar{s}, t)], \quad [\boldsymbol{\eta}, \bar{\boldsymbol{\lambda}}(s, \bar{s}, t)] = 0. \quad (5.30)$$

Note that $\mathbf{L}(s, \bar{s}, t)$ at $t = 0$ reduces to a generating function for bosonic products:

$$\mathbf{L}(s, \bar{s}, 0) = \sum_{n=0}^{\infty} (s\bar{s})^n \mathbf{L}_{n+1}^{(0,0)}. \quad (5.31)$$

Following the argument given in section 3.2, this boundary condition together with the differential equations (5.28)-(5.30) imply

$$[\mathbf{L}(s, \bar{s}, t), \mathbf{L}(s, \bar{s}, t)] = 0, \quad [\boldsymbol{\eta}, \mathbf{L}(s, \bar{s}, t)] = 0, \quad [\bar{\boldsymbol{\eta}}, \mathbf{L}(s, \bar{s}, t)] = 0. \quad (5.32)$$

Evaluating this at $s = \bar{s} = 0$ implies that the closed superstring products are in the small Hilbert space and satisfy the L_{∞} relations.

Now we have to solve (5.28)-(5.30) to define the products. Expanding (5.28) in powers gives the formula

$$\mathbf{L}_{n+2}^{(p,q)} = \frac{1}{p+q} \sum_{k=0}^n \left(\sum_{r,s} [\mathbf{L}_{n-k+1}^{(r,s)}, \boldsymbol{\lambda}_{k+2}^{(p-r,q-s)}] + \sum_{r,s} [\mathbf{L}_{n-k+1}^{(r,s)}, \bar{\boldsymbol{\lambda}}_{k+2}^{(p-r,q-s)}] \right). \quad (5.33)$$

The sum over r, s include all values such that the product and gauge product in the commutator have admissible picture numbers. Explicitly, in the commutator with $\boldsymbol{\lambda}$,

$$\begin{aligned} \sup(0, p-k-1) &\leq r \leq \inf(n-k, p-1), \\ \sup(0, q-k-1) &\leq s \leq \inf(n-k, q), \end{aligned} \quad (5.34)$$

and in the commutator with $\bar{\lambda}$,

$$\begin{aligned}\sup(0, p - k - 1) &\leq r \leq \inf(n - k, p), \\ \sup(0, q - k - 1) &\leq s \leq \inf(n - k, q - 1).\end{aligned}\tag{5.35}$$

Similar to (3.46), this formula determines the products recursively given the products of the bosonic string and the left/right gauge products. The left/right gauge products are defined by solving (5.29) and (5.30), and following the argument of section 3.2 we find natural solutions

$$\lambda_{n+2}^{(p+1,q)} = \frac{n-p+1}{n+3} \left(\xi_0 L_{n+2}^{(p,q)} - L_{n+2}^{(p,q)} (\xi_0 \wedge \mathbb{I}_{N+1}) \right), \tag{5.36}$$

$$\bar{\lambda}_{n+2}^{(p,q+1)} = \frac{n-q+1}{n+3} \left(\bar{\xi}_0 L_{n+2}^{(p,q)} - L_{n+2}^{(p,q)} (\bar{\xi}_0 \wedge \mathbb{I}_{N+1}) \right). \tag{5.37}$$

Once we know all products and gauge products with up to $n+1$ inputs, we can determine the $(n+2)$ nd superstring product $L_{n+2}^{(n+1,n+1)}$ by filling a “diamond” of products of intermediate picture number, starting from the bosonic product $L_{n+2}^{(0,0)}$ at the bottom. Filling the diamond requires climbing $4(n+1)$ levels, $2(n+1)$ of those require computing gauge products from products using (5.36) and (5.37), and the other $2(n+1)$ require computing products from gauge products using (5.33).

Just to see this work, let’s write the necessary formulas to determine the 3-product $L_3^{(2,2)}$, corresponding to the “diamond” sketched in 5.1. Start from the bosonic product:

$$\mathbf{L}_3^{(0,0)} = \text{given.} \tag{5.38}$$

In the first level we have two gauge products,

$$\lambda_3^{(1,0)} = \frac{1}{2} \left(\xi_0 L_3^{(0,0)} - L_3^{(0,0)} (\xi_0 \wedge \mathbb{I}_2) \right), \tag{5.39}$$

$$\bar{\lambda}_3^{(0,1)} = \frac{1}{2} \left(\bar{\xi}_0 L_3^{(0,0)} - L_3^{(0,0)} (\bar{\xi}_0 \wedge \mathbb{I}_2) \right). \tag{5.40}$$

In the second level, two products:

$$\mathbf{L}_3^{(1,0)} = [\mathbf{Q}, \lambda_3^{(1,0)}] + [\mathbf{L}_2^{(0,0)}, \lambda_2^{(1,0)}], \tag{5.41}$$

$$\mathbf{L}_3^{(0,1)} = [\mathbf{Q}, \lambda_3^{(0,1)}] + [\mathbf{L}_2^{(0,0)}, \lambda_2^{(0,1)}]. \tag{5.42}$$

In the third level, four gauge products:

$$\lambda_3^{(2,0)} = \frac{1}{4} \left(\xi_0 L_3^{(1,0)} - L_3^{(1,0)} (\xi_0 \wedge \mathbb{I}_2) \right), \quad (5.43)$$

$$\bar{\lambda}_3^{(1,1)} = \frac{1}{2} \left(\bar{\xi}_0 L_3^{(1,0)} - L_3^{(1,0)} (\bar{\xi}_0 \wedge \mathbb{I}_2) \right), \quad (5.44)$$

$$\lambda_3^{(1,1)} = \frac{1}{2} \left(\xi_0 L_3^{(0,1)} - L_3^{(0,1)} (\xi_0 \wedge \mathbb{I}_2) \right), \quad (5.45)$$

$$\bar{\lambda}_3^{(0,2)} = \frac{1}{4} \left(\bar{\xi}_0 L_3^{(0,1)} - L_3^{(0,1)} (\bar{\xi}_0 \wedge \mathbb{I}_2) \right). \quad (5.46)$$

In the fourth level, three products:

$$\mathbf{L}_3^{(2,0)} = \frac{1}{2} \left([\mathbf{Q}, \lambda_3^{(2,0)}] + [\mathbf{L}_2^{(1,0)}, \lambda_2^{(1,0)}] \right), \quad (5.47)$$

$$\mathbf{L}_3^{(1,1)} = \frac{1}{2} \left([\mathbf{Q}, \lambda_3^{(1,1)} + \bar{\lambda}_3^{(1,1)}] + [\mathbf{L}_2^{(0,0)}, \lambda_2^{(1,1)} + \bar{\lambda}_2^{(1,1)}] + [\mathbf{L}_2^{(0,1)}, \lambda_2^{(1,0)}] + [\mathbf{L}_2^{(1,0)}, \bar{\lambda}_2^{(0,1)}] \right), \quad (5.48)$$

$$\mathbf{L}_3^{(0,2)} = \frac{1}{2} \left([\mathbf{Q}, \bar{\lambda}_3^{(0,2)}] + [\mathbf{L}_2^{(0,1)}, \bar{\lambda}_2^{(0,1)}] \right). \quad (5.49)$$

In the fifth level, four gauge products:

$$\bar{\lambda}_3^{(2,1)} = \frac{1}{4} \left(\bar{\xi}_0 L_3^{(2,0)} - L_3^{(2,0)} (\bar{\xi}_0 \wedge \mathbb{I}_2) \right), \quad (5.50)$$

$$\lambda_3^{(2,1)} = \frac{1}{2} \left(\xi_0 L_3^{(1,1)} - L_3^{(1,1)} (\xi_0 \wedge \mathbb{I}_2) \right), \quad (5.51)$$

$$\bar{\lambda}_3^{(1,2)} = \frac{1}{2} \left(\bar{\xi}_0 L_3^{(1,1)} - L_3^{(1,1)} (\bar{\xi}_0 \wedge \mathbb{I}_2) \right), \quad (5.52)$$

$$\lambda_3^{(1,2)} = \frac{1}{4} \left(\xi_0 L_3^{(0,2)} - L_3^{(0,2)} (\xi_0 \wedge \mathbb{I}_2) \right). \quad (5.53)$$

In the sixth level, two products:

$$\mathbf{L}_3^{(2,1)} = \frac{1}{3} \left([\mathbf{Q}, \lambda_3^{(2,1)} + \bar{\lambda}_3^{(2,1)}] + [\mathbf{L}_2^{(1,0)}, \lambda_2^{(1,1)} + \bar{\lambda}_2^{(1,1)}] + [\mathbf{L}_2^{(1,1)}, \lambda_2^{(1,0)}] \right), \quad (5.54)$$

$$\mathbf{L}_3^{(1,2)} = \frac{1}{3} \left([\mathbf{Q}, \lambda_3^{(1,2)} + \bar{\lambda}_3^{(1,2)}] + [\mathbf{L}_2^{(0,1)}, \lambda_2^{(1,1)} + \bar{\lambda}_2^{(1,1)}] + [\mathbf{L}_2^{(1,1)}, \bar{\lambda}_2^{(0,1)}] \right). \quad (5.55)$$

In the seventh level, two gauge products:

$$\lambda_3^{(2,2)} = \frac{1}{4} \left(\xi_0 L_3^{(1,2)} - L_3^{(1,2)} (\xi_0 \wedge \mathbb{I}_2) \right), \quad (5.56)$$

$$\bar{\lambda}_3^{(2,2)} = \frac{1}{4} \left(\bar{\xi}_0 L_3^{(2,1)} - L_3^{(2,1)} (\bar{\xi}_0 \wedge \mathbb{I}_2) \right). \quad (5.57)$$

Finally, at the eighth level:

$$\mathbf{L}_3^{(2,2)} = \frac{1}{4} \left([\mathbf{Q}, \lambda_3^{(2,2)} + \bar{\lambda}_3^{(2,2)}] + [\mathbf{L}_2^{(1,1)}, \lambda_2^{(1,1)} + \bar{\lambda}_2^{(1,1)}] \right), \quad (5.58)$$

which is the 3-product of the closed superstring.

Let us mention a few generalizations of this construction. Instead of (5.28), we could define the products using the differential equation

$$\frac{\partial}{\partial t} \mathbf{L}(s, \bar{s}, t) = \left[\mathbf{L}(s, \bar{s}, t), c \lambda(s, \bar{s}, t) + \bar{c} \bar{\lambda}(s, \bar{s}, t) \right], \quad (5.59)$$

for c, \bar{c} arbitrary constants, while keeping equations (5.29) and (5.30) the same. It turns out that this setup can be transformed into the previous one by rescaling $\lambda, \bar{\lambda}$ and s, \bar{s} . The resulting products are related by

$$L_{n+1}^{(p,q)} \text{ (derived from (5.59))} = c^p \bar{c}^q L_{n+1}^{(p,q)} \text{ (derived from (5.28))} \quad (5.60)$$

In particular, $L_{n+1}^{(n,n)}$ derived from (5.59) is related to $L_{n+1}^{(n,n)}$ derived from (5.28) by a trivial factor $(c\bar{c})^n$, which can be absorbed into a redefinition of the coupling constant. A more nontrivial generalization is to take c and \bar{c} to be functions of t . This can be understood as follows. The form of the generating functions (5.27) suggests that \mathbf{L}, λ and $\bar{\lambda}$ can be thought of as depending on a fourth variable \bar{t} , which counts the rank of “rightmoving” multiplication. However, since left and right multiplication is identified, t and \bar{t} are not independent variables, and in (5.27) we took $t = \bar{t}$. However, we can imagine a more general relation between t and \bar{t} where they are taken to be functions of an independent parameter τ . Then (5.28) is naturally generalized to

$$\frac{\partial}{\partial \tau} \mathbf{L}(s, \bar{s}, \tau) = \left[\mathbf{L}(s, \bar{s}, \tau), \frac{dt(\tau)}{d\tau} \lambda(s, \bar{s}, \tau) + \frac{d\bar{t}(\tau)}{d\tau} \bar{\lambda}(s, \bar{s}, \tau) \right]. \quad (5.61)$$

Note that the parameter τ does not (in general) count picture number, and the coefficients of a power series expansion of $\mathbf{L}(s, \bar{s}, \tau)$ are general coderivations describing superpositions of the products with different picture numbers. This makes it difficult to extract the definition of the products from the solution to this differential equation. One application of this setup, however, is that the superstring products described here and those described

in section 5.1 can be formulated in a common language. They follow from two different choices of curves in the t, \bar{t} plane:

$$\text{This section : } \quad t(\tau) = \tau, \quad \bar{t}(\tau) = \tau, \quad (5.62)$$

$$\text{Section 5.1 : } \quad \begin{cases} t(\tau) = \tau, & \bar{t}(\tau) = 0, & \tau \in [0, T], \\ t(\tau) = T, & \bar{t}(\tau) = \tau - T, & \tau \in [T, 2T]. \end{cases} \quad (5.63)$$

In the former case, the products follow by evaluating \mathbf{L} at $s = \bar{s} = 0$ and $\tau = T$ and expanding in powers of T , while in the latter case, they follow from evaluating \mathbf{L} at $s = \bar{s} = 0$ and $\tau = 2T$ and expanding in powers of T . This gives one possible avenue to the proof of gauge equivalence between the products derived here and in section 5.1.

6 Conclusion

In this paper we have constructed explicit actions for all NS superstring field theories in the small Hilbert space. Closely following the calculations of [18], one can show that they reproduce the correct 4-point amplitudes. Since these actions share the same algebraic structure as bosonic string field theory, relaxing the ghost number of the string field automatically gives a solution to the classical BV master equation. This is a small, but significant step towards the goal of providing an explicit computational and conceptual understanding of quantum superstring field theory. The next steps of this program include

- Incorporate the Ramond sector(s) so as to maintain a controlled solution to the classical BV master equation.
- Quantize the theory. Specifically determine the higher genus corrections to the tree-level action needed to ensure a solution to the quantum BV master equation.
- Understand how the vertices and propagators of classical or quantum superstring field theory provide a single cover of the supermoduli space of super-Riemann surfaces.
- Understand how this relates to formulations of superstring field theory in the large Hilbert space, which may ultimately be more fundamental.

Progress on these questions will not only help to assess whether superstring field theory can be a useful tool beyond tree level, but may provide valuable insights into the systematics of superstring perturbation theory.

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A Quartic Vertices

The recursive construction of the vertices ultimately defines the action in terms of X s, ξ s, and products of the bosonic string. However, it is not necessarily easy to derive an explicit expression for the action in this form. The coalgebra notation offers great notational efficiency in expressing the recursive definition of the products, but it does not directly display the cyclically inequivalent contributions to each vertex (or, for the case of the closed string, the symmetrically inequivalent contributions). To obtain the cyclically or symmetrically inequivalent contributions, one must solve the recursion to the relevant order, expand the multi-string product (term-by-term) into X s, ξ s, and bosonic products, and then place each term into the respective cyclic or symmetric equivalence class. This procedure quickly becomes impractical to implement by hand; for example, the NS-NS quartic vertex involves 91 symmetrically inequivalent contributions, though depending on the construction some terms may vanish or be related by left/right symmetry. However, we are able to execute the computation out to quartic order for the open and heterotic string. We present our results here. It is an important open question whether a more efficient or direct method for computing the vertices in this form can be obtained.

Since individual contributions to each vertex contain the ξ zero mode, it is convenient to write the action using the symplectic form in the large Hilbert space. This is related to the symplectic form in the small Hilbert space through the formula [18]

$$\langle \omega_L | (\mathbb{I} \otimes \xi)(L \otimes L) = \langle \omega |, \quad (\text{A.1})$$

where L is the trivial map from the small Hilbert space to η -closed elements in the large Hilbert space. The vertices in the action can be written in several equivalent forms, related by cyclicity or by η -exact contributions. We fix this redundancy by requiring that one factor of X in the vertex always appears multiplied by ξ , and if ξX acts on an external state, it always acts on the first input of the symplectic form. If ξX does not act on an external state, there is always a remaining ξ which can again be chosen to act on the first entry of the symplectic form. With these choices, the cubic vertex in the open superstring action is

$$\frac{1}{3} \omega \left(\Phi, M_2^{(1)}(\Phi, \Phi) \right) = \frac{1}{3} \omega_L \left(\xi X \Phi, M_2^{(0)}(\Phi, \Phi) \right). \quad (\text{A.2})$$

The quartic vertex takes the form:

$$\begin{aligned}
\frac{1}{4} \omega \left(\Phi, M_3^{(2)}(\Phi, \Phi, \Phi) \right) = & \\
& \frac{5}{36} \left[\omega_L \left(\xi X \Phi, M_2^{(0)}(\Phi, \xi M_2^{(0)}(\Phi, \Phi)) \right) + \omega_L \left(\xi X \Phi, M_2^{(0)}(\xi M_2^{(0)}(\Phi, \Phi), \Phi) \right) \right] \\
& + \frac{1}{144} \left[\omega_L \left(\xi X \Phi, M_2^{(0)}(\Phi, M_2^{(0)}(\xi \Phi, \Phi)) \right) + \omega_L \left(\xi X \Phi, M_2^{(0)}(\Phi, M_2^{(0)}(\Phi, \xi \Phi)) \right) \right. \\
& \quad \left. + \omega_L \left(\xi X \Phi, M_2^{(0)}(M_2^{(0)}(\xi \Phi, \Phi), \Phi) \right) + \omega_L \left(\xi X \Phi, M_2^{(0)}(M_2^{(0)}(\Phi, \xi \Phi), \Phi) \right) \right] \\
& + \frac{1}{36} \left[\omega_L \left(\xi \Phi, M_2^{(0)}(\Phi, \xi X M_2^{(0)}(\Phi, \Phi)) \right) + \omega_L \left(\xi \Phi, M_2^{(0)}(\xi X M_2^{(0)}(\Phi, \Phi), \Phi) \right) \right] \\
& + \frac{1}{16} \left[\omega_L \left(\xi X \Phi, M_2^{(0)}(M_2^{(0)}(\Phi, \Phi), \xi \Phi) \right) - \omega_L \left(\xi X \Phi, M_2^{(0)}(\xi \Phi, M_2^{(0)}(\Phi, \Phi)) \right) \right] \\
& + \frac{1}{16} \left[\omega_L \left(\xi X^2 \Phi, M_3^{(0)}(\Phi, \Phi, \Phi) \right) + \omega_L \left(\xi X \Phi, M_3^{(0)}(\Phi, X \Phi, \Phi) \right) \right] \\
& + \frac{1}{8} \left[\omega_L \left(\xi X \Phi, M_3^{(0)}(X \Phi, \Phi, \Phi) \right) \right]. \tag{A.3}
\end{aligned}$$

If $M_2^{(0)}$ is Witten's associative star product, we can set $M_3^{(0)} = 0$ and the quartic vertex simplifies to:

$$\begin{aligned}
\frac{1}{4} \omega \left(\Phi, M_3^{(2)}(\Phi, \Phi, \Phi) \right) = & \\
& \frac{5}{36} \left[\omega_L \left(\xi X \Phi, M_2^{(0)}(\Phi, \xi M_2^{(0)}(\Phi, \Phi)) \right) + \omega_L \left(\xi X \Phi, M_2^{(0)}(\xi M_2^{(0)}(\Phi, \Phi), \Phi) \right) \right] \\
& - \frac{1}{18} \left[\omega_L \left(\xi X \Phi, M_2^{(0)}(\Phi, M_2^{(0)}(\Phi, \xi \Phi)) \right) + \omega_L \left(\xi X \Phi, M_2^{(0)}(\xi \Phi, M_2^{(0)}(\Phi, \Phi)) \right) \right] \\
& + \frac{1}{36} \left[\omega_L \left(\xi \Phi, M_2^{(0)}(\Phi, \xi X M_2^{(0)}(\Phi, \Phi)) \right) + \omega_L \left(\xi \Phi, M_2^{(0)}(\xi X M_2^{(0)}(\Phi, \Phi), \Phi) \right) \right]. \tag{A.4}
\end{aligned}$$

The 3-vertex for the heterotic string is

$$\frac{1}{3!} \omega \left(\Phi, L_2^{(1)}(\Phi, \Phi) \right) = \frac{1}{3!} \omega_L \left(\xi_0 X_0 \Phi, L_2^{(0)}(\Phi, \Phi) \right), \tag{A.5}$$

and the 4-vertex is

$$\begin{aligned}
\frac{1}{4!} \omega \left(\Phi, L_3^{(2)}(\Phi, \Phi, \Phi) \right) = & \\
& \frac{5}{108} \omega_L \left(\xi_0 X_0 \Phi, L_2^{(0)}(\Phi, \xi_0 L_2^{(0)}(\Phi, \Phi)) \right) + \frac{1}{216} \omega_L \left(\xi_0 X_0 \Phi, L_2^{(0)}(\Phi, L_2^{(0)}(\Phi, \xi_0 \Phi)) \right) \\
& + \frac{1}{108} \omega_L \left(\xi_0 \Phi, L_2^{(0)}(\Phi, \xi_0 X_0 L_2^{(0)}(\Phi, \Phi)) \right) - \frac{1}{48} \omega_L \left(\xi_0 X_0 \Phi, L_2^{(0)}(\xi_0 \Phi, L_2^{(0)}(\Phi, \Phi)) \right) \\
& + \frac{1}{96} \omega_L \left(\xi_0 X_0^2 \Phi, L_3^{(0)}(\Phi, \Phi, \Phi) \right) + \frac{1}{32} \omega_L \left(\xi_0 X_0 \Phi, L_3^{(0)}(\Phi, \Phi, X_0 \Phi) \right). \tag{A.6}
\end{aligned}$$

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